# Countable subdirect powers of finite commutative semigroups

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# Subdirect products

**Definition**

A *subdirect product* of two semigroups $S$ and $T$ is a subsemigroup $U$ of the direct product $S \times T$ for which the projection maps

$$\pi_S : U \to S,\ (s,t) \mapsto s,$$
$$\pi_T : U \to T,\ (s,t) \mapsto t,$$

onto $S$ and $T$ are surjections.

# Examples of subdirect products

- The direct product $S \times T$ is a subdirect product of semigroups $S$ and $T$.
- $\Delta_S := \{(s,s) : s \in S\}$ is the diagonal subdirect product of a semigroup $S$ with itself.
- Let $F$ be the group with presentation

$$\left\langle x, y \mid [xy^{-1}, x^{-1}yx] = [xy^{-1}, x^{-2}yx^2] = 1 \right\rangle .$$

Then $\left\langle (x, y^{-1}), (y, x), (x^{-1}, x^{-1}), (y^{-1}, y) \right\rangle$ is a subdirect product of $F$ with itself, which is not equal to $F \times F$ or $\Delta_F$.

# Subdirect powers

Recall that for a family of sets $\{X_i\}_{i \in I}$ for some infinite indexing set $I$, the infinite Cartesian product is defined

$$\prod_{i \in I} X_i := \{f : I \to \cup_{i \in I} X_i \mid (\forall i \in I)(f(i) \in X_i)\}.$$

It will be easier to view the elements of a countable Cartesian product of a family of sets as sets of countable strings in the following way;

$$\prod_{i \in \mathbb{N}} X_i = \{x_1 x_2 x_3 \ldots : x_i \in X_i \text{ for } i \in \mathbb{N}\}.$$

The symbol $x_1$ will be called the "first component" of the string $x = x_1 x_2 x_3 \ldots$, $x_2$ will be the "second component", and so on. In general, the $i$-th component of a countable string $x$ will be denoted $[x]_i$.

# Subdirect powers

If the sets $X_i$ are all equal to the same set $S$, we will instead refer to the countable Cartesian product as a *countable Cartesian power*, denoted $S^{\mathbb{N}}$.

**Exercise:** A countable direct power of a semigroup $S$ is a (possibly uncountably) infinite semigroup $S^{\mathbb{N}}$, with componentwise multiplication

$$s_1 s_2 s_3 \ldots \cdot t_1 t_2 t_3 \ldots = (s_1 \cdot t_1)(s_2 \cdot t_2)(s_3 \cdot t_3) \ldots$$

A *subdirect power* of a semigroup $S$ is a subsemigroup of $S^{\mathbb{N}}$ for which the projection maps onto each component are surjections.

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# Subdirect powers of finite groups

**Theorem - McKenzie (1982)**

A finite group $G$ has countably many non-isomorphic countable subdirect powers if and only if $G$ is abelian.

We'd like to work towards analogous results for subdirect powers of finite semigroups, that look like

**Theorem**

A finite semigroup $S$ has countably many non-isomorphic countable subdirect powers if and only if $S$ satisfies *insert fascinating semigroup properties here*.

# Subdirect powers of finite groups

For this talk, we will concentrate on finite **commutative** semigroups.

## Definition

A finite commutative semigroup $S$ will be called *countable type* if it has only countably many non-isomorphic countable subdirect powers, and *uncountable type* if it has uncountably many such.

# Some small examples

Firstly, the trivial semigroup of course is countable type, because $\{1\}^{\mathbb{N}} \cong \{1\}$.

The commutative semigroups of order $2$ up to isomorphism are $\mathbb{Z}_2$ (countable type), $O_2$ (countable type) and $U_1 = \{0, 1\}$, the two element semilattice.

$U_1$ can be viewed as a linearly ordered set with the natural ordering $0 \leq 1$. Moreover, any subdirect power of $U_1$ will also be a semilattice, and can similarly be considered as an ordered set via $x \leq y \iff x \wedge y = x$.

# The case for $U_1$

A quick side definition:

**Definition**

For a finite word $u \in S^+$, we will denote by $\overline{u}$ the infinite string

$$uuu... \in S^{\mathbb{N}}.$$

An element $v$ of $S^{\mathbb{N}}$ is said to be *recurring* if $v = \overline{u}$ for some finite word $u \in S^+$. Similarly, a subsemigroup of $S^{\mathbb{N}}$ is said to be recurring if all of its elements are recurring.

# The case for $U_1$

**Lemma**

For two recurring elements $s, t \in U_1^{\mathbb{N}}$ with $s \leq t$, there exists $u \in U_1^{\mathbb{N}}$ with $u \neq s$, $u \neq t$, but $s \leq u \leq t$.

**Lemma**

$U_1^{\mathbb{N}}$ contains an order isomorphic copy of $\mathbb{Q}$, consisting of recurring elements .

**Lemma**

$U_1^{\mathbb{N}}$ contains uncountably many linear orders consisting of recurring elements, up to order isomorphism.

# The case for $U_1$

Two linear orders are order isomorphic if and only if they are isomorphic as semilattices, and so this gives us uncountably many recurring subsemilattices of $U_1^{\mathbb{N}}$.

A subdirect product can be constructed from each one, and any two non-isomorphic semilattices will give non-isomorphic subdirect products with this construction.

This gives

**Proposition**

$U_1$ is of uncountable type.

# Semilattices

We can exploit $U_1$ to show that all (non-trivial) semilattices are also going to be of uncountable type as well, via the following framework.

If $Y$ is a finite non-trivial semilattice with operation $\wedge$, then $Y$ has a least element $0$ and a minimal idempotent $e \neq 0$ with respect to the ordering on $Y$ induced by $\wedge$.

$\{0, e\} \cong U_1$, from which we can make uncountably many non-isomorphic countable recurring subdirect powers $S$. For any of these, it is easy to construct a countable recurring subdirect power of $Y$ by "adding in the diagonal"

$$\overline{Y} = \{\overline{y} \in Y^{\mathbb{N}} : y \in Y\}$$

# Semilattices

Two non-isomorphic recurring subdirect powers of $U_1$ will also give two non-isomorphic recurring subdirect powers of $Y$ in this construction, and hence

**Theorem**

Any non-trivial semilattice $Y$ is of uncountable type.

# Semigroups with more than one idempotent

Take a finite commutative semigroup $S$ with $E(S) > 1$, and let $L$ be the greatest semilattice homomorphic image of $S$. Note that $L = S/\eta$, where $\eta$ is the congruence on $S$ defined

$$(a, b) \in \eta \Leftrightarrow (\exists x, y \in S^1)(\exists m, n \in \mathbb{N})(ax = b^m)(by = a^n).$$

Further, the congruence classes of $\eta$ (that is, the elements of $L$) are precisely the Archimedean components of $S$.

Again, we can construct uncountably many recurring subdirect powers of $L$, from each of which we can construct a countable recurring subdirect power of $S$ by making all possible "string replacements" by elements of the Archimedian component.

# Semigroups with more than one idempotent

The greatest semilattice homomorphic image of the resulting subdirect power of $S$ will be exactly the subdirect power of $L$ you started with, which is an isomorphic invariant. Hence...

**Theorem**

Any finite commutative semigroup $S$ wih $E(S) > 1$ is of uncountable type.

# Semigroups with a unique idempotent

**Lemma**

Let $S$ be a finite commutative semigroup with a unique idempotent. Then $S$ is either a group, or an ideal extension of a group by a $k$-nilpotent semigroup.

The case where $S$ is a group has been dealt with. So it remains to consider ideal extensions of groups by $k$-nilpotent semigroups. We will start with $k$-nilpotent semigroups, noting that $k = 2$ has been dealt with.

# $k$-nilpotent semigroups

Let $S$ be a finite $k$-nilpotent commutative semigroup with $k \geq 3$, and let $M \subseteq \mathbb{N} \setminus \{1, 2, \ldots, |S|\}$ be an infinite subset. Fix an element $x \in S^{k-1} \setminus \{0\}$ with $d(x)$ minimal. Fix also an element $y \in S$ such that $y \mid x$.

Let

$$X = \left\{ (0)^{p-1} s \overline{0} \in S^{\mathbb{N}} : p \in \mathbb{N}, s \in S \right\},$$

$$Y_M = \bigcup_{p \in \mathbb{N}} \left\{ (0)^{p-1} y (x)^q \overline{0} \in S^{\mathbb{N}} : 1 \leq q \leq m_p \right\},$$

where $m_p$ is the $p$-th element of $M$, when ordered in the natural way. Define

$$\mathcal{S}(S, x, y; M) := \langle X \cup Y_M \rangle \leq S^{\mathbb{N}}.$$

# $k$-nilpotent semigroups

**Lemma**

For a $k$-nilpotent semigroup $S$ with $k \geq 3$, and $x, y \in S$ as above, if $M, N \subseteq \mathbb{N} \setminus \{1, \ldots, |S|\}$ with $M \neq N$, then

$$\mathcal{S}(S, x, y; M) \ncong \mathcal{S}(S, x, y; N)$$

**Corollary**

Finite commutative $k$-nilpotent semigroups are of uncountable type.

# Extensions of $k$-nilpotent semigroups

The last cases to consider are ideal extensions of groups by $k$-nilpotent semigroups where the group is non-trivial, splitting into two cases: $k = 2, k \geq 3$.

There are similar complicated constructions for countable subdirect powers of these as above for the $k$-nilpotent case which I won't cover, but we have...

**Corollary**

Ideal extensions of non-trivial groups by $k$-nilpotent semigroups for $k \geq 2$ are of uncountable type.

# Final classification

**Theorem (C, Ruškuc, 2021)**

A finite commutative semigroup $S$ is of countable type if and only if $S$ is either a group, or a null semigroup.

# Further questions

What are the types of non-commutative completely simple semigroups? What about finite semigroups in general?

Thank you for listening!